Quotient Groups and Homomorphism Theorems

(Section 2.7)

Recall: If G, H are groups, $\varphi : G \to H$ is a homomorphism, then $\ker(\varphi) \lhd G$. We know that the converse is true, but how to prove it?

The quotient group construction gives us exactly the target we need!

Theorem: (quotient group construction)

Let G be a group, $H \trianglelefteq G$.

Consider G/H to be

the left cosets of H in G,

under the multiplication

$$(xH)\cdot(yH) = xyH$$

$\forall\ x, y \in G$. Then

G/H is a group.

proof: Only one thing to really check:

is this operation well-defined?

What this means:

Suppose

$$x_1 H = x_2 H$$

and $y_1 H = y_2 H$.

Is it the case that

$$x_1 y_1 H = x_2 y_2 H \text{ ?}$$

To see that this is the case, use the proposition from last class: $x_1 y_1 H = x_2 y_2 H$

if and only if

$$(x_2 y_2)^{-1} x_1 y_1 \in H.$$

But we know

$$x_1 H = x_2 H \text{, so}$$

$$x_2^{-1} x_1 \in H.$$

$$(x_2 y_2)^{-1} x_1 y_1$$

$$= y_2^{-1} \underbrace{x_2^{-1} x_1}_{\in H} y_1$$

We also know

$$y_1 H = y_2 H \text{, so}$$

$$y_1 y_2^{-1} = k \in H$$

Then $y_2^{-1} = y_1^{-1} k$, so

$$(x_2 y_2)^{-1} x_1 y_1$$

$$= y_2^{-1} \underbrace{x_2^{-1} x_1}_{\in H} y_1$$

$$= (y_1^{-1} k) x_2^{-1} x_1 y_1$$

$$= y_1^{-1} (\underbrace{k}_{\in H} \underbrace{x_2^{-1} x_1}_{\in H}) y_1$$

We know that $H \leq G \Rightarrow$

$k x_2^{-1} x_1 \in H$.

Since H is normal, w

know

$$y_1^{-1} H y_1 = H$$

Therefore,

$$(x_2 y_2)^{-1} x_1 y_1 = y_1^{-1} (k x_2^{-1} x_1) y_1 \in y_1^{-1} H y_1 = H,$$

and so

$$x_2 y_2 H = x_1 y_1 H.$$ ✓

Now check the rest of the group axioms!

( ) Identity of G/H is $H = eH$, since $\forall\ x \in G$,

$$xH \cdot H = xH \cdot eH$$
$$= (x \cdot e) H = xH.$$

Similarly, $H \cdot (xH) = xH$,

so we have that $eH = H$

is the identity of G/H.

2) Inverse of xH in G/H is

$x^{-1}H$, since

$$(xH) \cdot (x^{-1}H) = (x \cdot x^{-1})H$$

$$= eH = H.$$

Similarly,

$(x^{-1}H)(xH) = H$, so

$(xH)^{-1} = x^{-1}H$.

3) associativity: let $x, y, z \in G$.

$$xH \cdot (yH \cdot zH)$$

$$= xH(y \cdot z\, H)$$

$$= x(y \cdot z)\, H$$

$$= (x \cdot y) \cdot z\, H \quad \text{(associativity in } G\text{)}$$

$$= (x \cdot y)H \cdot zH$$

$$= (x \cdot H \cdot yH) \cdot zH$$

So G/H is a group! $\square$

Corollary: (normal subgroups are kernels)

Let G be a group, $H \trianglelefteq G$.

Then $\exists$ a group K and

a homomorphism $\varphi : G \to K$

with $\boxed{\ker(\varphi) = H}$

proof: Let $K = G/H$. Define

$\varphi : G \to K$

$\varphi(x) = xH$.

If $y \in G$, then

$$\varphi(x\cdot y) = (x\cdot y)H$$
$$= (xH)\cdot(yH)$$
$$= \varphi(x)\cdot\varphi(y)$$

So φ is a homomorphism!

By the proposition from last class, $x \in \ker(\varphi) = \{y \in G \mid \varphi(y) = H\}$ if and only if $xH = H$, i.e., $x \in H$. Therefore, $\ker(\varphi) = H$.

$\square$